Fall 2016

EMISSARY

Mathematical Sciences Research Institute

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Dehn Filling and Negative Curvature

Henry Wilton

This semester, MSRI is hosting a program in geometric group theory. Since there is only space for a whistle-stop tour of the subject, we will confine ourselves to some recent developments that trace their origins back to the work of Max Dehn a century ago.

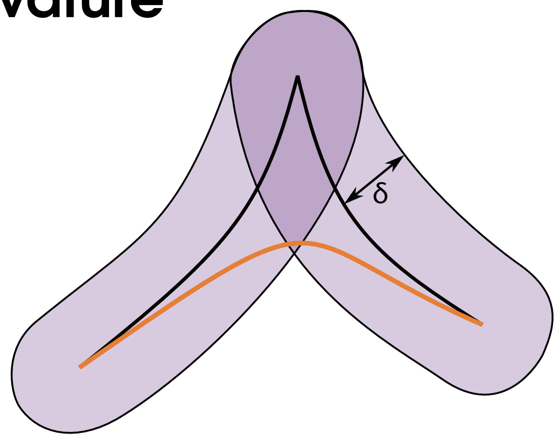
In 1912, Dehn set out a template for the field of geometric group theory when he used the geometry of the hyperbolic plane $\mathbb{H}^2$ to solve the *word problem* in the fundamental group of a hyperbolic surface Σ. The word problem for a group Γ asks for an effective procedure (nowadays we would say an *algorithm*) to decide whether a given element of Γ is trivial. The fundamental group of the surface, $\pi_1\Sigma$, consists of homotopy classes of loops on Σ, and so a solution to the word problem would give an effective procedure to decide which loops on Σ can be continuously deformed to a point.

Dehn's strategy is a powerful one: to study a group Γ (in this case $\pi_1\Sigma$), we should understand the geometry of some space on which Γ acts (in this case, the universal cover of Σ, which can be identified with $\mathbb{H}^2$). Every element g of $\pi_1\Sigma$ is represented by a loop γ on Σ, and Dehn's solution proceeds by lifting γ to $\mathbb{H}^2$ and applying hyperbolic geometry. This interplay between geometry, algebra, and topology that was a feature of his work is now characteristic of the field.

As well as using negatively curved spaces to study algebraic properties of groups, another of his innovations was the use of what is now called *Dehn filling* to construct new examples of 3-manifolds. As we shall see, far-reaching developments of Dehn's techniques continue to give us a geometric understanding of a multitude of groups and spaces.

Hyperbolic Groups

Dehn's use of hyperbolic geometry to solve the word problem in $\pi_1\Sigma$ is extremely elegant, but was inherently limited to fundamental groups of negatively curved manifolds. Modern geometric group theory was kick-started in the 1980s by Gromov's definition of *coarse* negative curvature.



A δ-slim triangle. (Image courtesy of Samuel Brown.)

Consider a triangle Δ, with vertices x, y, z, in a geodesic metric space X; that is, Δ is the union of three geodesics, $[x, y]$, $[y, z]$, and $[z, x]$. For $\delta \geqslant 0$, we say that a triangle is called δ-*slim* if every point on every side of Δ is within a distance δ of one of the other two sides. (See the figure above.)

> Let $\delta \geqslant 0$. A geodesic metric space X is δ-*hyperbolic* if every triangle in X is δ-slim.

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Rikki Ward Photo

What We All Need to Know About Physics: Read about Sir Roger Penrose in conversation with David Eisenbud at the Commonwealth Club of San Francisco, and also look for your invitation to the **2017 National Math Festival** in the Public Understanding Highlights on page 9.

David Eisenbud

The View from MSRI

David Eisenbud, Director

MSRI has become fundamental to the world's mathematical research enterprise, and it is thriving. Here are some of the latest signs.

Science

I was delighted, as I studied the applications to be a Research Professor (RP) at MSRI in 2016–17, to learn that fully 40% more people applied this year than at any time in the last 10 years. Moreover, the field is very deep — the applications look stronger than ever.

The RPs are distinguished mathematicians who are 10 or more years past the Ph.D. and stay at MSRI for at least three months. They serve as mentors to our postdoctoral fellows, as organizers of seminars, and often as guides to what's important in the fields of the programs. The presence of such members is a crucial component of MSRI's success.

We don't have to wait for next year; there's intense scientific activity here and now. Here's a sample:

1) **Geometric Group Theory** currently fills our whole building — a "jumbo" program in the local org-speak. It too had an unusually large number of applicants, and is going very well. The picture above shows a meeting of the program's weekly lunch seminar.

2) Continuing a 10-year-old collaboration with the **Chicago Mercantile Exchange**, we have just chosen Robert Wilson, of Stanford, as the winner of the 2016 CME Group–MSRI Prize for Quantitative Applications in Finance and Economics. Wilson is known among other things for the introduction of game theory into economics. I remember that when the prize was established, the committee informally proposed to target people who could win the Nobel, but hadn't yet. As I write, Bengt Holmstrom has just become the fifth Nobel prize winner in the group. Coincidentally, there were also five Nobelists on the 2016 selection committee.

3) In a new collaboration with the Howard Hughes Medical Institute (HHMI), we'll run a workshop on **Insect Navigation** (How do mosquitoes find you?), bringing together mathematical, physical, and neurophysiological aspects, at HHMI's Janelia Research Campus, near Washington, DC, from December 7–9.

4) Another first for us: we are proud to announce **postdoctoral fellowships** named for **Dusa McDuff** and **Karen Uhlenbeck**, two exceptional mathematicians who are long-time members of the MSRI family (most recently as Trustees). A biography of Dusa McDuff, as well as the inaugural McDuff Postdoctoral Fellow, David Hume, is featured on page 3. The Spring 2017 issue of this newsletter will feature a profile of Karen Uhlenbeck as well as our first Uhlenbeck Postdoctoral Fellow.

Public Understanding of Mathematics

Lots is happening in this domain as well; here are some novel items:

1) The **Navajo Math Circles Project** that was featured in MSRI's 2016 documentary by George Csicsery (see page 9) will expand into other parts of the Navajo Nation and other Native American tribes. The program has already had significant effects: for example, the tribal college has seen a marked increase in the demand for mathematics courses and has just enrolled 15 students in its first degree program in mathematics education. On the strength of this success the Carnegie Corporation of New York has provided MSRI with funds to continue the work and extend it to other tribes.

2) **Unusual Kingdoms**: Who doesn't know Prokofiev's *Peter and the Wolf*? It turns out that there's a shortage of — and a hunger for — pieces for professional and school orchestras to play at children's concerts. As far as I know, there weren't *any* with any math content. Now there is one: *Peace Comes to the Unusual Kingdoms* has a libretto by MSRI Trustee Jeff Goodby (if you were in the advertising world you'd know his name from "Got Milk?" and a host of other campaigns) and music by Jack Perla. Its world premiere this month by the Oakland Symphony will be led by MSRI Trustee Maestro Michael Morgan. Could this be our most unusual project?

3) The first ever **National Math Festival** was organized by MSRI with the collaboration of the Institute for Advanced Study (IAS) in Washington last year. It was spread over several Smithsonian museums but it outgrew the venue within the first hour: some 25,000 people showed up, some from as far away as Massachusetts. Come, if you can, to the second National Math Festival, which will take place on April 22, 2017! We're organizing it this time in collaboration with the IAS and with MoMath, the National Museum of Mathematics, and it will be in the Washington Convention Center. Check out nationalmathfestival.org as well as the article on page 9 to preview some of the festival. You will also find links to interesting math activities for the public — feel free to suggest more!

Sustainability

At 35, MSRI is doing well, and still evolving in its service to the mathematical community. The model consists of scientifically focused programs involving a broad group of mathematical leaders as well as young researchers.

A board of distinguished mathematicians has the responsibility for choosing and guiding the programs, and — since they serve only four-year terms — the board's view of mathematics is constantly refreshed. MSRI's beautiful home, high on a ridge that overlooks San Francisco and the Bay in the hills of Berkeley, is iconic. Though initially viewed as an experiment, MSRI is now a well-proven, nationally and internationally appreciated part of the world's mathematical infrastructure.

Will MSRI Be Here in 50 Years?

Like the permanence of the IAS, whose different model is in many ways complementary, the continued vitality of MSRI is of great value for the mathematical community. MSRI's mode of convening critical masses around one mathematical topic after another, combined with an ideal of accessibility and community support, has been highly effective and widely emulated.

But continuity cannot be taken for granted. The NSF has been a faithful supporter since MSRI's beginning, but institutes do sometimes disappear. Moreover, NSF funding is projected at the 2010 level through 2020, keeping up neither with inflation nor with the increasing demand for MSRI's programs.

For MSRI to fulfill its mission of support for the mathematical community over the next 50 years and beyond, private funding will be increasingly necessary. Only a significant endowment can ensure stability and allow for modest growth and flexibility. To prepare for building such an endowment, MSRI's trustees have launched a study of MSRI's past and present development effort, to determine what works well and what must be strengthened for continued excellence and impact.

The New McDuff Fellowship

MSRI provides unique opportunities for postdoctoral fellows by bringing them together with leading researchers in their fields in an environment that promotes creativity and the effective interchange of ideas and techniques. Thirty-two postdoctoral fellows spent a semester or more at MSRI during academic year 2015–16, and six of them were supported by privately funded fellowships.

Dusa McDuff

An anonymous donor has made possible two new named fellowships at MSRI, created in honor of Dr. Dusa McDuff and Dr. Karen Uhlenbeck, both of whom are current MSRI trustees. The first of the new fellowships to begin in 2016–17 is the McDuff Fellowship, awarded this semester to David Hume (also profiled here).

Dusa McDuff is an internationally renowned mathematician who continues to make seminal contributions to the area of symplectic geometry. In 1991, the American Mathematical Society awarded her the prestigious Ruth Lyttle Satter Prize, and more recently the London Mathematical Society (LMS) awarded her the 2010 Senior Berwick Prize in recognition of two of her papers published the previous year. Dusa was also the first female Hardy Lecturer of the LMS. She has been an inspiration for generations of mathematicians.

Dusa has also been an important figure at MSRI for more than 25 years. A member of MSRI's Symplectic Geometry program in 1988, she joined the Scientific Advisory Committee in 1990 and became its chair in 1993. She chaired the Board of Trustees from 1998–2001 and has been a Trustee ever since. In that time she led the Board in the movement toward the major building renovation and addition that was completed in 2006. This was MSRI's first significant fundraising effort, and her understanding of the needs and possibilities was crucial.

Dusa received her Ph.D. from the University of Cambridge in 1971 and has been in the United States since 1978, initially as an Assistant Professor at Stony Brook University. She is currently the Helen Lyttle Kimmel '42 Professor of Mathematics at Barnard College. Among her numerous honors, she was elected a Fellow of the Royal Society in London in 1994 and a Fellow of the American Academy of Arts and Sciences in 1995. She became a member of the United States National Academy of Sciences in 1999.

. . . and First McDuff Postdoc

David Hume is the inaugural recipient of the new Dusa McDuff Fellowship this fall. David obtained his M.Sci. at the University of Birmingham in 2009, then his Ph.D. at Oxford in 2013 (under the supervision of Cornelia Druțu).

David Hume

After a short postdoctoral stay at Bar-Ilan (Israel) in 2013, he spent two years at the Université Catholique de Louvain in Belgium, then one year in 2015–2016 at Université Paris-Sud in the Laboratoire de Mathématiques d'Orsay. After his semester at MSRI, David will return to Oxford for a 3-year position as Titchmarsh Research Fellow.

David has already established himself as an expert in asymptotic and coarse geometry of finitely generated groups. Among his results, the following is especially striking and simple (to state, not to prove!) : Up to coarse equivalence, there exists a continuum of families of expander graphs, all having unbounded girth. The proof cleverly appeals to the separation profile, introduced for infinite graphs by Benjamini, Schramm, and Timar.

GGT Program — Dehn Filling and Negative Curvature

(continued from page 1)

It is a nice exercise to use the fact that triangles in $\mathbb{H}^2$ have bounded area to check that $\mathbb{H}^2$ is δ-hyperbolic for some δ. (Hint: consider an inscribed semicircle.) The next examples are trees equipped with the natural path-length metric, which are easily seen to be 0-hyperbolic.

Just as Dehn studied the fundamental groups of hyperbolic surfaces via their geometric actions on the hyperbolic plane, so we may study the class of groups that act geometrically on δ-hyperbolic spaces.

> A group Γ is *hyperbolic* if it acts properly discontinuously by isometries on a proper δ-hyperbolic metric space X with compact quotient.

The choice of the space X turns out to be unimportant — Γ must be finitely generated, and we may take X to be a Cayley graph for Γ with respect to any finite generating set.

Since $\mathbb{H}^2$ is δ-hyperbolic, it follows that fundamental groups of closed hyperbolic surfaces (and, indeed, hyperbolic n-manifolds) are also hyperbolic, justifying the terminology. The actions of free non-abelian groups on their (0-hyperbolic) Cayley trees shows they too are hyperbolic.

Gromov's insight was that almost everything that works for fundamental groups of hyperbolic manifolds can be generalized to hyperbolic groups. Indeed, he was able to show that the hyperbolic groups are *precisely* those that Dehn's solution to the word problem for surfaces applies to. Gromov was also able to provide strong evidence that, as is the case for 2- and 3-dimensional manifolds, hyperbolicity is the generic behavior in group theory, by demonstrating that a suitably defined "random group" is hyperbolic.

Dehn Filling

Another early achievement of Dehn was to extend Poincaré's construction of his famous homology sphere to an infinite family of examples. The construction he used, now called *Dehn filling*, takes knots in the 3-sphere as a starting point for building new manifolds.

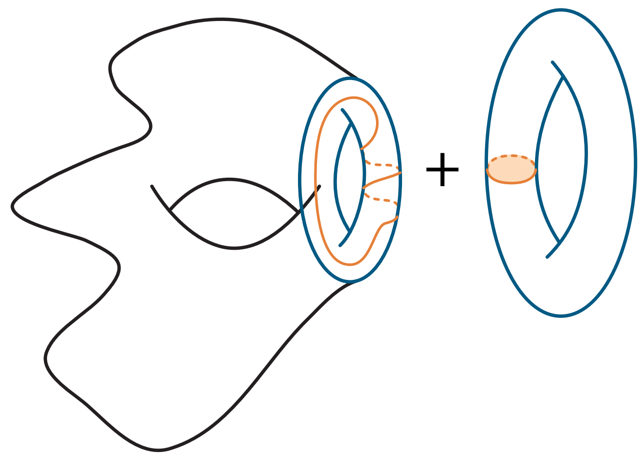
Consider, for instance, a compact 3-manifold M with boundary ∂M homeomorphic to the 2-torus (the complement of a tubular neighborhood of a knot in the 3-sphere is a typical example). The set of embedded, essential circles in ∂M are called *slopes*, and can be identified with the rational numbers $\mathbb{Q}$. To fill M, one chooses a slope p/q and glues the boundary of a solid torus $T = D \times S^1$ to ∂M in such a way that the boundary of the meridional disc D is identified with the slope p/q. The resulting manifold $M(p/q)$ is called a *Dehn filling* of M. (See the figure at top right.)

As well as building new examples, Dehn filling can also be used to study the original manifold M. For instance, the complement of the unknot has the property that every Dehn filling is a lens space, so if we want to prove that a certain knot K is non-trivial, one approach is to find a Dehn filling of the complement of K that is not a lens space.

A finer understanding of Dehn fillings was initiated by Thurston, who was able to understand their geometry with his *hyperbolic Dehn filling theorem*:

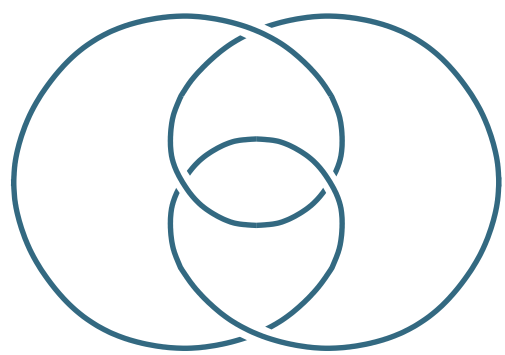
> If the interior of M is hyperbolic, then for all but finitely many slopes p/q, the Dehn filling $M(p/q)$ is also hyperbolic.

Combined with Thurston's discovery that the complements of many knots carry hyperbolic structures — the simplest example is the figure-eight knot — he was able to demonstrate the extraordinary abundance of hyperbolic 3-manifolds.



A schematic picture of Dehn filling. The glued tori are blue, and the slope p/q and meridional disc D are orange. (Image courtesy of Dave Futer.)

From the point of view of geometric group theory, we can ask what effect Dehn filling has on the fundamental group $\Gamma = \pi_1 M$. The boundary component ∂M corresponds, up to conjugacy, to a subgroup P of Γ. When the interior of M is hyperbolic, it has the special property that its boundary ∂M is *acylindrical* — every nontrivial map of an annulus A into M, with both components of ∂A mapping into ∂M, can be homotoped into ∂M (holding ∂A fixed).



The complement of the figure-eight knot has a hyperbolic structure. (Image courtesy of Saul Schleimer.)

This translates into a group-theoretic condition on P: whenever any conjugate gPg^{-1} intersects P non-trivially, it follows that $g \in P$.

Focus on the Scientist: Ian Agol

Ian Agol is one of the organizers of this semester's Geometric Group Theory program. He grew up in Walnut Creek, California, not far from his current employer, the University of California, Berkeley. Most of his work has been in the geometry and topology of three-manifolds. Ian is responsible for some of the most spectacular recent applications of geometric group theory, proving a conjecture of Dani Wise which in turn resolved the virtual Haken and virtual fibering questions in the topology of hyperbolic three-manifolds.

Ian Agol

Wise's conjecture was that hyperbolic cubulated groups always admit (virtual) embeddings into right-angled Artin groups. Hyperbolic groups generalize fundamental groups of closed hyperbolic three-manifolds, known to be cubulated from the work of Kahn–Markovic, Sageev, and others. Even though the virtual Haken conjecture was about three-manifolds (and can be phrased in terms of their fundamental groups), it was essential to Ian's approach to work in the more general category of hyperbolic groups. Many of the key insights involved transporting tools from the study of three-manifolds to this setting.

Ian is a gregarious mathematician, an active participant on MathOverflow, and was one of the first mathematics bloggers. The mathematics community is very important to Ian; in addition to this semester's MSRI program he organized a special year (2015–2016) in three-manifolds at the Institute for Advanced Study. Ian has been a strong advocate for more open and lower-priced journals, publishing much of his most important work in journals like the *Journal of Topology* (formed after the editorial board of *Topology* defected from Elsevier) or *Documenta Mathematica* (a completely free journal founded by the German Mathematical Society).

For his contributions, Ian was awarded the 2009 Clay Research Award (for the Marden and Ahlfors measure conjectures, jointly awarded with Calegari and Gabai), the 2013 Veblen Prize (jointly with Dani Wise), and the 2016 Breakthrough Prize in Mathematics. Other honors include Simons, Sloan, Guggenheim, and Miller fellowships and the Senior Berwick Prize. Ian is a member of the National Academy of Sciences and a Fellow of the American Mathematical Society.

— Jason Manning

The condition says that the subgroup P is as far as possible from being normal, and hence such subgroups are called *malnormal*.

From the group-theoretic perspective, the operation of Dehn filling kills the subgroup of P generated by the slope p/q. Let us introduce some terminology that makes it convenient to state the conclusion of the Dehn filling theorem: the phrase *almost all* subgroups N of P means all subgroups N disjoint from S, for some finite subset S of $P \smallsetminus \{1\}$. We can then state the group-theoretic version of Thurston's Dehn filling theorem as follows:

> For almost all cyclic summands N of P, the quotient $\Gamma/\langle\langle N\rangle\rangle$ is infinite and hyperbolic.

(Here, the notation $\langle\langle N\rangle\rangle$ denotes the normal closure of N in Γ, the smallest normal subgroup of Γ that contains N.)

One of the most remarkable recent developments in the theory of hyperbolic groups is the discovery that Dehn filling can be massively generalized to the world of groups. There are several instances of this, but the following *combinatorial Dehn filling theorem* of Groves–Manning and Osin is the most direct generalization of the statement for 3-manifolds:

> Let Γ be a hyperbolic group, and P a malnormal, quasiconvex subgroup. For almost all subgroups $N \lhd P$, if P/N is hyperbolic then the quotient $\Gamma/\langle\langle N\rangle\rangle$ is infinite and hyperbolic.

As a side note: *Quasiconvexity* is a technical hypothesis on P, which enables us to take advantage of the hyperbolic geometry of the group Γ. It can be thought of as analogous to the definition of a *geometrically finite* subgroup in the classical theory of hyperbolic manifolds. We have omitted mention of the theory of *relatively hyperbolic groups*, which plays an important role in this story. If hyperbolic groups generalize the fundamental groups of closed hyperbolic manifolds, relatively hyperbolic groups generalize the fundamental groups of hyperbolic 3-manifolds with acylindrical boundary.

It is worth pausing for a moment to consider how hugely the scope of the original theorem has been expanded by passing to group theory, since the combinatorial Dehn filling theorem gives new information even in the case when M is a hyperbolic 3-manifold and P is the subgroup corresponding to an essential surface Σ in M. In classical Dehn filling, one can only fill Σ if it is a subsurface of the boundary, since this is required for the result to remain a manifold. The theorem of Groves–Manning and Osin applies to *any* geometrically finite surface Σ whose fundamental group is malnormal. To be sure, the result of Dehn filling is no longer a manifold, but as a hyperbolic group, it has many useful features that one can exploit to study M and Σ, just as studying the Dehn fillings of a knot provides useful information about that knot.

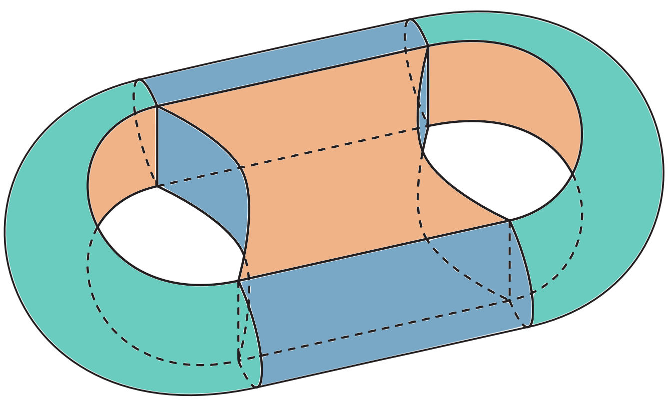
Combinatorial Dehn filling is an example of the power of the point of view of geometric group theory, and as we shall see, it can be used to prove deep theorems even in the world of 3-manifolds that inspired it. In the remainder of this article, we shall mention two improved versions of combinatorial Dehn filling, and describe their applications.

Special Cube Complexes

One reason that hyperbolic groups are so useful is that they are easy to construct (as evidenced by Gromov's theorem that a random group is hyperbolic). However, there are many situations in which we would like flexible constructions of *linear groups* (that is, groups of matrices). Failing that, we would often like our groups to be *residually finite*, meaning that the intersection of their finite-index subgroups is trivial. (A straightforward algebraic argument using congruence quotients shows that finitely generated linear groups are residually finite.)

Unfortunately, there are examples of nonlinear hyperbolic groups, and it is a famous open question whether every hyperbolic group is residually finite, so the construction of linear groups needs a new insight. Perhaps surprisingly, the first really flexible such construction came not from algebra, but from geometry, with Haglund–Wise's definition of *special cube complexes*.

As the name suggests, a cube complex is a cell complex in which every n-dimensional cell is an n-dimensional cube, and the gluing maps respect the natural cubical structure on the boundaries of the cells. We will always take our cube complexes X to be *non-positively curved*, meaning that the links of the vertices are flag complexes. In this case, there is a tight connection between the geometry of X and its fundamental group $\pi_1 X$.



A non-positively curved cube complex homeomorphic to the surface of genus 2. (Image courtesy of Samuel Brown.)

A cube complex may seem like a rather constrained and rigid structure, but a technique of Sageev, called *cubulation*, provides a powerful tool for showing that a manifold M is homotopy-equivalent to a cube complex. An n-dimensional cube C has a natural collection of n codimension-one hypercubes, each of which cuts C in half in a different direction. In a cube complex X, these piece together to form *(immersed) hyperplanes*, which together determine the structure of the cube complex. Sageev and Bergeron–Wise showed that a sufficiently rich set of codimension-one submanifolds $N_1, \ldots, N_n$ immersed in M is naturally *dual* to a non-positively curved cube complex X, with the same fundamental group as M.

Haglund–Wise called a non-positively curved cube complex *special* if its hyperplanes did not exhibit one of four pathologies: one-sidedness, self-intersection, self-osculation, and inter-osculation. These pathologies are all quite straightforward, and are illustrated in the figure on the facing page. The fundamental group of a special cube complex is called a *special group*, and as is very often the case in geometric group theory, it is more natural to consider *virtually special* groups, meaning those groups with a special subgroup of finite index.

Haglund and Wise showed that special cube complexes enjoy a wide variety of extremely useful properties. A flavor of these is given by the following:

> If a cube complex X is special, then $\pi_1 X$ admits a faithful embedding into $GL_n(\mathbb{Z})$ for some n.

An illustration of the power and flexibility of their construction is that, combined with earlier work of Bridson–Miller, it followed quickly that the isomorphism problem is unsolvable for finitely presented linear groups. In a variety of collaborations, Wise went on to provide many other beautiful constructions of special cube complexes, but most remarkably of all, he was able to show that it is possible to perform combinatorial Dehn filling while staying inside the world of virtually special groups.

His conclusion is slightly weaker than the conclusion of our previous Dehn filling theorems, and we therefore need a new definition. Let P be a group. We say that a property holds *for a positive fraction* of the normal subgroups $N \lhd P$ if there is a subgroup Q of finite index in P, and the property holds for all N contained in Q. With this terminology, Wise's *malnormal special quotient theorem* states:

> Let Γ be a hyperbolic, virtually special group, and P a malnormal, quasiconvex subgroup. For a positive fraction of normal subgroups $N \lhd P$, the quotient $\bar{\Gamma} := \Gamma / \langle\langle N \rangle\rangle$ is hyperbolic and virtually special.

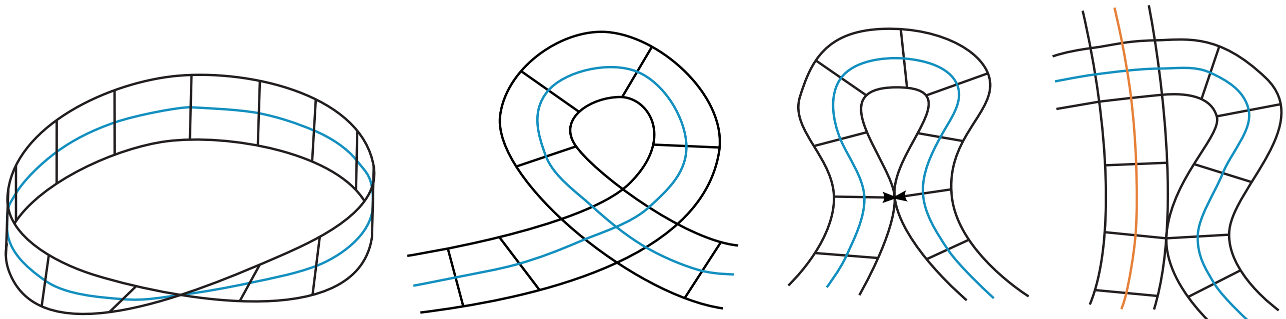
It is hard to overstate how surprising this theorem is: starting with a virtually special (in particular, linear) group Γ, Wise finds a rich family of linear quotients $\bar{\Gamma}$. In general, there is no reason for a quotient of a linear group to remain linear.

The Virtually Haken Theorem

Perelman's famous *geometrization theorem* was an epochal breakthrough in the study of 3-manifolds, providing a complete classification of their geometry, analogous to the *uniformization theorem* for Riemann surfaces. But it was not the end of the story: although it tells us that generic 3-manifolds are hyperbolic, geometrization leaves untouched the question of a topological description of hyperbolic 3-manifolds. Perhaps the most straightforward such description was the *virtually Haken conjecture*, which posits:

> Every closed, hyperbolic 3-manifold M has a finite-sheeted covering space that contains an essential closed surface.

The virtually Haken conjecture can be conveniently broken into two parts: first, that every hyperbolic 3-manifold M contains an essential, immersed, closed surface Σ; and second, an immersed surface in M can be lifted to an embedding in a finite-sheeted covering space.



A special cube complex may not exhibit any of these pathologies. (Image courtesy of Samuel Brown.)

The first part was resolved by Kahn–Markovic, who used techniques from ergodic theory to build a a rich family of essential, immersed surfaces in an arbitrary hyperbolic 3-manifold. Crucially, the families of surfaces that they exhibit are rich enough for Sageev's construction to apply, and as a corollary one obtains that every hyperbolic 3-manifold is homotopy equivalent to a non-positively curved cube complex.

The second part was resolved by Agol, who proved the following much more general conjecture of Wise:

> If a group Γ is hyperbolic and cubulated, then Γ is virtually special.

It is not hard to see how the virtually Haken conjecture follows. By the work of Kahn–Markovic, a hyperbolic 3-manifold M is homotopy equivalent to non-positively curved cube complex X. Agol's theorem applies to the fundamental group of X, and so X has a finite-sheeted, special covering space X_0, which in turn corresponds to a covering space M_0 of M. Since X_0 is special, its hyperplanes are embedded. We may choose a homotopy equivalence $M_0 \to X_0$ which is transverse to a hyperplane, and the preimage of that hyperplane is then an embedded surface in M_0, which we may compress until it becomes essential.

The proof of Agol's theorem is too complex to discuss here, but the malnormal special quotient theorem plays a crucial role. Despite the fact that the virtually Haken theorem is a statement about manifolds, the proof uses the geometry of hyperbolic groups in an essential way.

Acylindrically Hyperbolic Groups

The family of hyperbolic groups has been a rich vein for geometric group theory, but of course many groups of natural interest are not hyperbolic — the simplest constraint is provided by the fact that hyperbolic groups cannot contain subgroups isomorphic to $\mathbb{Z}^2$. These considerations motivated Osin to define the class of acylindrically hyperbolic groups.

Recall that a group Γ is hyperbolic if it acts properly discontinuously and cocompactly by isometries on a hyperbolic space X. Acylindrically hyperbolic groups are defined in the same way, but the hypothesis of proper discontinuity is weakened, and instead we only assume that the action of the group Γ on the hyperbolic space X is *acylindrical*. Roughly, this means that coarse stabilizers of points sufficiently far apart are bounded. The precise definition of an acylindrical action is as follows:

> For every $\epsilon > 0$ there exist R, N such that whenever $x, y \in X$ are at distance at least R, the set
> $$\{\gamma \in \Gamma \mid d(x, \gamma.x), d(y, \gamma.y) < \epsilon\}$$
> has cardinality at most N.

A group Γ is then defined to be *acylindrically hyperbolic* if it acts acylindrically on some δ-hyperbolic space. (We also insist that the action is *non-elementary*, which rules out degenerate actions, such as an action on a point.)

This definition dramatically increases the size of the class of groups considered. Indeed, hyperbolic groups are necessarily finitely presented, so there are only countably many, whereas acylindrically hyperbolic groups need not even be finitely generated, and there turn out to be uncountably many of them.

The first nontrivial (in the sense of neither hyperbolic nor relatively hyperbolic) example is provided by $\mathrm{Mod}(\Sigma)$, the mapping class group of a surface Σ. If we choose disjoint essential simple closed curves α, β on Σ, then the Dehn twists in α and β are easily seen to generate a copy of $\mathbb{Z}^2$, so $\mathrm{Mod}(\Sigma)$ is not hyperbolic, and a finer analysis shows that $\mathrm{Mod}(\Sigma)$ is also not relatively hyperbolic. However, there is a natural hyperbolic space on which $\mathrm{Mod}(\Sigma)$ acts: its complex of curves.

The *curve complex* $C(\Sigma)$ is a simplicial complex naturally associated to Σ. The 0-simplices are isotopy classes of essential simple closed curves, and a set of curves $\alpha_0, \ldots, \alpha_n$ spans an n-simplex if and only if they can be isotoped on Σ to be mutually disjoint. It's clear that $\mathrm{Mod}(\Sigma)$ acts naturally on $C(\Sigma)$; moreover, by deep theorems of Masur–Minsky and Bowditch, $C(\Sigma)$ is δ-hyperbolic, and the action is acylindrical.

Many similarly natural classes of groups also turn out to be acylindrically hyperbolic: outer automorphism groups of free groups, the fundamental groups of non-geometric 3-manifolds, certain Artin and Coxeter groups, and so on. The definition appears to be particularly well adapted to transformation groups of naturally occurring objects, and perhaps the most intriguing examples in this vein are provided by *Cremona groups*.

Given a field k, the *Cremona group in n-variables*, $\mathrm{Bir}(\mathbb{P}^n(k))$, is the group of birational transformations of the projective space $\mathbb{P}^n(k)$, or, equivalently, the group of k-automorphisms of the field $k(x_1, \ldots, k_n)$. For $n = 1$ this coincides with the familiar group

GGT Program *(continued from previous page)*

$\mathrm{PGL}_2(k)$, and so the first new examples arise when $n = 2$. These groups are of great interest in algebraic geometry, but as infinitely generated groups their study had seemed quite inaccessible to the techniques of geometric group theory. This changed dramatically when Cantat–Lamy were able to show that $\mathrm{Bir}(\mathbb{P}^2(\mathbb{C}))$ acts naturally on an infinite-dimensional hyperbolic space $\mathbb{H}_{\bar{\mathcal{Z}}}$, and it follows from their study of this action that $\mathrm{Bir}(\mathbb{P}^2(\mathbb{C}))$ is acylindrically hyperbolic.

Of course, for such an expanded class of groups, one can only expect to prove weaker theorems. It therefore comes as some surprise to discover that combinatorial Dehn filling results can be extended into this domain.

In order to Dehn fill a group Γ, one needs a suitably "nice" subgroup P, with controlled geometry — for hyperbolic Γ, we required P to be malnormal and quasiconvex. Dahmani–Guirardel–Osin defined a *hyperbolically* embedded subgroup P of an arbitrary group Γ, and Osin showed that the existence of a proper, infinite, hyperbolically embedded subgroup is equivalent to Γ's being acylindrically hyperbolic. Most remarkably, Dahmani–Guirardel–Osin proved a combinatorial Dehn filling theorem for hyperbolically embedded subgroups:

> If a subgroup P is hyperbolically embedded in a group Γ, then for almost all normal subgroups $N \lhd P$, the quotient $\Gamma/\langle\langle N \rangle\rangle$ is acylindrically hyperbolic.

This theorem provides a highly non-obvious source of infinite quotients of Γ — indeed, one can deduce that acylindrically hyperbolic groups have uncountably many distinct normal subgroups. Most intriguingly, the normal subgroups that one finds in this way are often inaccessible to algebraic techniques.

Again, perhaps the most striking application is to the Cremona group $\mathrm{Bir}(\mathbb{P}^2(\mathbb{C}))$. Enriques had asked in 1896 whether the Cremona group is simple. Cantat–Lamy's construction of an action on a hyperbolic space answered this question and, on the contrary, produced a very rich supply of normal subgroups.

In conclusion, acylindrical hyperbolicity enables us to apply techniques developed in hyperbolic geometry to wider problems in algebra and geometry. Even the most unlikely groups may admit an action on a hyperbolic space, and we can learn a great deal about a group if we can find that action!

Gamelin Postdoc

Michael Cantrell is this fall's Gamelin Endowed Postdoctoral Fellow as a member of the Geometric Group Theory program.

Michael Cantrell

Michael received his undergraduate degree from Yale University in 2009. In July of 2016, he obtained his Ph.D. from the University of Illinois at Chicago under the supervision of Alex Furman. Starting in January 2017, Michael will be an Assistant Professor at the University of Michigan.

Michael works at the interface of ergodic theory and geometric group theory. He has studied the Cayley graph of a nilpotent finitely generated group for which the length of all the edges is chosen at random in a translation-invariant way. He has proved that, seen from far away, such a graph has an asymptotic limit that is a Carnot–Caratheodory metric space. A key ingredient of the proof is an extension of Kingman's subadditive ergodic theorem to subcocycles on nilpotent groups.

The Gamelin postdoctoral fellowship was created in 2014 by Dr. Ted Gamelin, Emeritus Professor of the UCLA Department of Mathematics. The Gamelin Fellowship emphasizes the important role that research mathematicians play in the discourse of K-12 education.

Viterbi Postdoc

Eugenia (Jenya) Sapir is the Viterbi Endowed Postdoctoral Fellow this fall in the Geometric Group Theory program.

Jenya Sapir

Jenya received her B.S. with honors in Mathematics at the University of Chicago in 2008. She received a Fulbright U.S. Student Scholarship for study at ENS Lyon in Lyon, France in 2008–2009. She then received an NSF Graduate Research Fellowship 2009–2014 for graduate study. She received her Ph.D. under the supervision of Maryam Mirzakhani in 2014 at Stanford University. She has been the J. L. Doob Assistant Professor at the University of Illinois at Urbana–Champaign since 2014.

Jenya works in the field of geometric topology. She is particularly interested in counting closed curves on a hyperbolic surface. In her most striking work she has found upper and lower bounds for the number of closed curves with K self intersections and of length at most some number L.

The Viterbi postdoctoral scholarship is funded by a generous endowment from Dr. Andrew Viterbi, well known as the co-inventor of Code Division Multiple Access based digital cellular technology and the Viterbi decoding algorithm, used in many digital communication systems.

Highlights: Public Understanding of Mathematics

Your Invitation! — 2017 National Math Festival

Saturday, April 22, 2017
Washington, DC
nationalmathfestival.org

If you're in DC this April, we hope you'll join us for the National Math Festival — organized by MSRI in cooperation with the **Institute for Advanced Study** (IAS) and the **National Museum of Mathematics** (MoMath). The first National Math Festival was held in Washington, DC, in April 2015 and attracted over 20,000 attendees.

This free, public event features a full-day program of lectures by some of the most influential mathematicians of our time with hands-on demonstrations, art, films, performances, puzzles, games, children's book readings, and more to showcase the unexpected sides of mathematics for all ages. In addition to the festivities in the nation's capital, giant SOMA Cube workshops will be held at science museums around the U.S.

MSRI is proud to join the American Mathematical Society (AMS), the Association of Women in Mathematics (AWM), the Mathematical Association of America (MAA), and the Society for Industrial and Applied Mathematics (SIAM) in promoting public understanding of mathematics through programs including Mathematics Awareness Month (mathaware.org), the online resources available at More Math! (nationalmathfestival.org/more-math), and the National Math Festival.

Festival Presenters — More to Be Announced!

American Mathematical Society • Dr. Stephon Alexander (Brown University) • Dr. Arthur Benjamin (Harvey Mudd College) • The Bridges Organization • Dr. Alissa S. Crans (Loyola Marymount University) • Dr. Robbert Dijkgraaf (Institute for Advanced Study) • Dr. Maria Droujkova (Natural Math) • Dr. Marcus du Sautoy (University of Oxford) • Elwyn and Jennifer Berlekamp Foundation • *FIRST* • First 8 Studios at WBGH • Gathering 4 Gardner • Paul Giganti (California Math Festival Program) • Dr. Herbert Ginsburg (Columbia University) • Dr. Rebecca Goldin (George Mason University) • Brady Haran (Numberphile) • Dr. George Hart (Bridges Organization) • Elisabeth Heathfield (Making Math Visible) • Julia Robinson Mathematics Festival • Dr. Poh-Shen Loh (Carnegie Mellon University) • Dr. Talea L. Mayo (University of Central Florida) • National Museum of Mathematics • Natural Math • NOVA and NOVA Labs • Dr. Stephanie Palmer (University of Chicago) • Matt Parker (Queen Mary University of London) • Science Cheerleader • Dr. Cliff Stoll • Dr. James Tanton (Mathematical Association of America) • Dr. Richard Tapia (Rice University) • ThinkFun • Dr. Mariel Vazquez (University of California, Davis) • The Young People's Project • Dr. Mary Lou Zeeman (Bowdoin University)

National Association of Math Circles

The National Association of Math Circles (NAMC) continues to offer much-needed resources and support to the Math Circle community. NAMC recently hosted its second three-day training workshop for the Math Circle–Mentorship and Partnership (MC–MAP) Program. Over 50 mathematicians and educators gathered from around the country to participate in the workshop, aimed at training 13 novice Math Circle leadership teams on the academic and administrative components of leading Math Circles.

On October 28–30, "Circle on the Road" was held in New York City in partnership with the Center for Mathematical Talent at the Courant Institute of the Mathematical Sciences at New York University. To receive periodic updates on NAMC events, create an account at mathcircles.org or visit facebook.com/mathcircles.

Sir Roger Penrose in Conversation

On October 4, 2016, MSRI partnered with the Commonwealth Club of San Francisco to host a conversation between renowned theoretical physicist Sir Roger Penrose and MSRI director David Eisenbud entitled "What We All Need to Know About Physics" (see cover photo). Penrose's new book, *Fashion, Faith, and Fantasy in the New Physics of the Universe* formed the basis of the conversation at the sold-out event, echoing his 2006 public address in honor of MSRI's Chern Hall building celebration events.

Navajo Math Circles

MSRI's newest documentary with director George Csicsery, Navajo Math Circles, has been screening at film festivals and events around the United States, including the United Nations Association Film Festival (California), the Bluff Arts Festival (Utah), the Laughlin International Film Festival (Nevada), as well as the national meetings of SACNAS, the American Indian Science and Engineering Society, and in many other cities. The film's television premiere on PBS in September 2016 was part of Spotlight Education Week, and it continues to be broadcast on PBS channels. For more information, visit pbs.org/program/navajo-math-circles/.

A New MSRI Face

Sandra Peterson joined MSRI in March as the new Development Assistant. Prior to that, she worked as a database manager for development at the San Francisco Zen Center. Sandra enjoys traveling, photography, and hiking in the Bay Area with her dog, Tashi.

Focus on the Scientist: Ruth Charney

Ruth Charney is the Theodore and Evelyn Berenson Professor of Mathematics at Brandeis University. She has made important and influential contributions to a wide range of subjects in group theory and topology. These include seminal work on homological stability for linear groups, contributions to understanding the stable cohomology of mapping class groups and compactifications of moduli spaces, work on strict hyperbolization of cell complexes, work on the topology and geometry of non-positively curved (that is, locally CAT(0)) spaces, and work developing the theory of Artin groups, especially right-angled Artin groups and their automorphisms.

Ruth Charney

One particularly beautiful theorem, with Mike Davis, shows that many infinite-type Artin groups satisfy the classical $K(\pi, 1)$ conjecture; Deligne proved this for finite-type Artin groups but Charney and Davis use completely different methods based on CAT(0) geometry. Ruth continues to produce first-rate mathematics; most recently she introduced a new type of boundary for CAT(0) spaces that has aroused a great deal of interest and found several striking applications.

On a personal level, Ruth is extremely approachable, gives clear and compelling lectures, and interacts readily with mathematicians of all ages; in particular, she has been a very successful mentor of graduate students and postdocs.

Ruth also has an exemplary record of service to the mathematical community. A small sample of her past service includes the editorial board of Algebraic and Geometric Topology, the board of trustees of MSRI, vice president of the American Mathematical Society and president of the Association for Women in Mathematics. She is currently a member of the board of trustees of the American Mathematical Society.

Ruth received her Ph.D. from Princeton and subsequently held postdocs at the University of California, Berkeley and Yale University. She accepted a permanent position at the Ohio State University in 1984, then moved to Brandeis in 2003. She has also held visiting research positions at ETH Zurich, the Institut Mittag-Leffler in Sweden, the Institute for Advanced Study in Princeton, the Mathematical Institute in Oxford, the Institut des Hautes Études Scientifiques in Paris, and the Université de Bourgogne in Dijon.

— Karen Vogtmann

Call for Membership

MSRI invites membership applications for the 2017–2018 academic year in these positions:

Research Members	by December 1, 2016
Postdoctoral Fellows	by December 1, 2016

In the academic year 2017–2018, the research programs are:

Geometric Functional Analysis and Applications
Aug 14–Dec 15, 2017
Organized by Franck Barthe, Marianna Csornyei, Boaz Klartag, Alexander Koldobsky, Rafal Latala, Mark Rudelson

Geometric and Topological Combinatorics
Aug 14–Dec 15, 2017
Organized by Jesus De Loera, Vic Reiner, Francisco Santos, Francis Su, Rekha Thomas, Günter M. Ziegler

Group Representation Theory and Applications
Jan 16–May 25, 2018
Organized by Robert Guralnick, Alexander (Sasha) Kleshchev, Gunter Malle, Gabriel Navarro, Julia Pevtsova, Raphael Rouquier, Pham Tiep

Enumerative Geometry Beyond Numbers
Jan 16–May 25, 2018
Organized by Mina Aganagic, Denis Auroux, Jim Bryan, Andrei Okounkov, Balazs Szendroi

MSRI uses **MathJobs** to process applications for its positions. Interested candidates must apply online at mathjobs.org. For more information about any of the programs, please see msri.org/scientific/programs.

Forthcoming Workshops

Dec 6–9, 2016: *Amenability, Coarse Embeddability and Fixed Point Properties*

Dec 7–9, 2016: *Insect Navigation*

Jan 19–20, 2017: *Connections for Women: Harmonic Analysis*

Jan 23–27, 2017: *Introductory Workshop: Harmonic Analysis*

Feb 2–3, 2017: *Connections for Women: Analytic Number Theory*

Feb 6–10, 2017: *Introductory Workshop: Analytic Number Theory*

Mar 15–17, 2017: *CIME 2017: Observing for Access, Power, and Participation in Mathematics Classrooms as a Strategy to Improve Mathematics Teaching and Learning*

Mar 27–31, 2017: *Hot Topics: Galois Theory of Periods and Applications*

May 1–5, 2017: *Recent Developments in Analytic Number Theory*

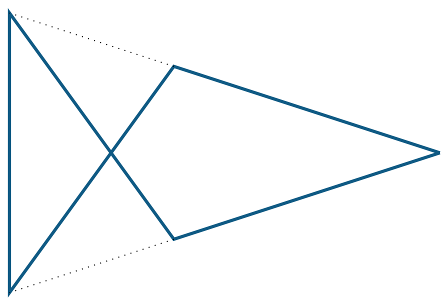
May 15–19, 2017: *Recent Developments in Harmonic Analysis*

For more information about any of these workshops, as well as a full list of all upcoming workshops and programs, please see msri.org/scientific.

Puzzles Column

Elwyn Berlekamp and Joe P. Buhler

1. Five identical matchsticks form the curious figure below. What is the angle between the rightmost two matchsticks?

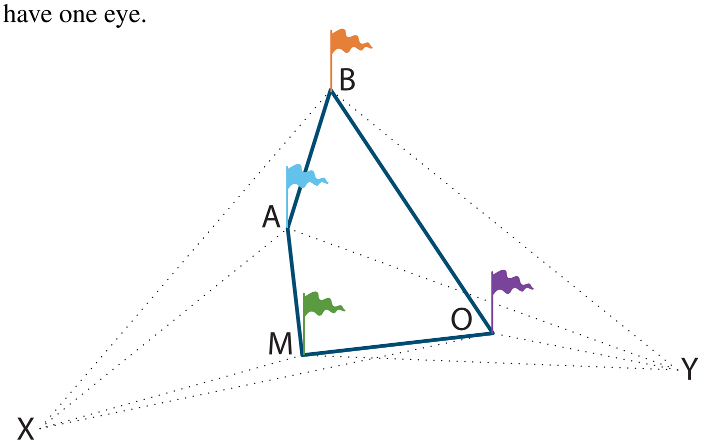


Comment: This is the very first of the "Dionigmas," due to Dion Gijswijt, in the recent MAA book *Half a Century of Pythagoras Magazine*. The book captures some of the best bits of a charming Dutch magazine aimed at talented high school math students. The chapter containing Gijswijt's enigmas is especially interesting, as these Dionigmas often seem to lack sufficient information to allow a solution.

2. If $S \subset \mathbb{R}^3$ is a set of points in 3-space then let $L(S)$ denote the set of all points that are on a line that joins distinct points in S. For instance if T is the set of four vertices of a tetrahedron then $L(T)$ is the union of six lines obtained by extending the edges of the tetrahedron. What is $L(L(T))$?

Comment: Apparently this is an old chestnut due to Victor Klee.

3. (a) The corners of a convex quadrilateral with no two sides parallel are labeled with the letters B, A, M, O. What is the maximum number of distinct four-letter words that can be seen by observers who are outside the quadrilateral? For instance, in the figure BAMO is visible from X and MOAB is visible from Y. In the idealized world in which this puzzle lives, the surface of the earth is an infinite flat plane, the letters are on top of thin vertical posts which can be seen from an arbitrary distance, and all observers have one eye.



(b) Consider the same question about a convex n-gon whose vertices are labeled with n distinct letters: What is the largest possible number of n-letter words (strings of distinct letters) than can be seen by an observer outside the convex n-gon?

Comment: Question **(b)** was the last problem on the 2016 BAMO exam.

Call for Proposals

All proposals can be submitted to the Director or Deputy Director or any member of the Scientific Advisory Committee with a copy to proposals@msri.org. For detailed information, please see the website msri.org.

Thematic Programs

The Scientific Advisory Committee (SAC) of the Institute meets in January, May, and November each year to consider letters of intent, pre-proposals, and proposals for programs. The deadlines to submit proposals of any kind for review by the SAC are **March 15**, **October 15**, and **December 15**. Successful proposals are usually developed from the pre-proposal in a collaborative process between the proposers, the Directorate, and the SAC, and may be considered at more than one meeting of the SAC before selection. For complete details, see tinyurl.com/msri-progprop.

Hot Topics Workshops

Each year MSRI runs a week-long workshop on some area of intense mathematical activity chosen the previous fall. Proposals should be received by **March 15**, **October 15**, or **December 15** for review at the upcoming SAC meeting. See tinyurl.com/msri-htw.

Summer Graduate Schools

Every summer MSRI organizes several 2-week long summer graduate workshops, most of which are held at MSRI. Proposals must be submitted by **March 15**, **October 15** or **December 15** for review at the upcoming SAC meeting. See tinyurl.com/msri-sgs.

Named Positions for Fall 2016

Eisenbud and Simons Professors

Mladen Bestvina, University of Utah
Ruth Charney, Brandeis University
Koji Fujiwara, Kyoto University
Howard Masur, University of Chicago
Kasra Rafi, University of Toronto
Zlil Sela, Hebrew University
Alain Valette, Université de Neuchâtel

Named Postdoctoral Fellows

Gamelin: Michael Cantrell, University of Illinois at Chicago
McDuff: David Hume, Université Paris-Sud
Viterbi: Eugenia (Jenya) Sapir, University of Illinois at Urbana–Champaign

Clay Senior Scholar

Karen Vogtmann, University of Warwick

MSRI is grateful for the generous support that comes from endowments and annual gifts that support faculty and postdoc members of its programs each semester.

The Clay Mathematics Institute awards its Senior Scholar awards to support established mathematicians to play a leading role in a topical program at an institute or university away from their home institution.

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Add @*msri.org* to email addresses. All area codes are (510) unless otherwise noted.

Scientific and Education

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Hélène Barcelo, Deputy Director, 643-6040, *deputy.director*
Diana White, National Association of Math Circles Director, 303-315-1720, *namc.director*

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Kirsten Bohl, Project Lead, National Math Festival, 499-5181, *kbohl*
Arthur Bossé, Operations Manager, 643-8321, *abosse*
Patricia Brody, Housing Advisor, 643-6468, *pbrody*
Heike Friedman, Director of Development, 643-6056, *hfriedman*
Aaron Hale, IT Strategist and Technical Lead, 643-6069, *ahale*
Mark Howard, Facilities and Administrative Coordinator, 642-0144, *mhoward*
Tracy Huang, Assistant for Scientific Activities, 643-6467, *thuang*
Claude Ibrahimoff, International Scholar Advisor & Executive Assistant, 643-6019, *cibrahimoff*
Lisa Jacobs, Director's Assistant and Board Liaison, 642-8226, *lisaj*
Christine Marshall, Program Manager, 642-0555, *chris*
Jinky Rizalyn Mayodong, Staff Accountant, 642-9798, *rizalyn*
Jennifer Murawski, Communications and Event Coordinator, 642-0771, *jmurawski*
Megan Nguyen, Grants Specialist, 643-6855, *megan*
Sandra Peterson, Development Assistant, 642-0448, *speterson*
Linda Riewe, Librarian, 643-1716, *linda*
Jacari Scott, Scholar Services Coordinator, 642-0143, *jscott*
Sanjani Varkey, Family Services Consultant, *sanjani*
Stefanie Yurus, Controller, 642-9238, *syurus*

— Join us! —

For two events
at the **2017 JMM**
in Atlanta

Mathematical Institutes Open House
Wednesday, January 4
5:30–8:00 pm
International Ballroom
Int Tower LL1

MSRI Reception for Current and Future Donors
Thursday, January 5
6:30–8:00pm
University Room
Conference Level

(Both events in the Hyatt Regency Atlanta)